

Asteroseismology of the GW Virginis stars SDSS J0349–0059 and VV 47

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ABSTRACT

Context. GW Virginis stars are a well-studied class of non-radial g -mode pulsators. SDSS J0349–0059 and VV 47 are two PG 1159 stars members of this class of variable stars. SDSS J0349–0059 is an interesting GW Vir star that shows a complete pulsation spectrum, that includes rotational splitting of some of its frequencies. VV 47 is a pulsating PG 1159 star surrounded by a planetary nebula. This star is particularly interesting because it exhibits a rich and complex pulsation spectrum.

Aims. We present an asteroseismological study of SDSS J0349–0059 and VV 47 aimed mainly at deriving their total mass on the basis of state-of-the-art PG 1159 evolutionary models.

Methods. We compute adiabatic nonradial g -mode pulsation periods for PG 1159 evolutionary models with stellar masses ranging from 0.515 to 0.741 M_{\odot} , that take into account the complete evolution of the progenitor stars. We first estimate a mean period spacing for both SDSS J0349–0059 and VV 47, and then we constrain the stellar mass of these stars by comparing the observed period spacing with the asymptotic period spacing and with the average of the computed period spacings. We also employ the individual observed periods to search for a representative seismological model for each star. Finally, we estimate the rotation period of SDSS J0349–0059.

Results. We found a spectroscopic mass of $M_{\star} \sim 0.543 M_{\odot}$ for SDSS J0349–0059 and $M_{\star} \sim 0.529 M_{\odot}$ for VV 47. By comparing the observed period spacing with the asymptotic period spacing we obtain $M_{\star} \sim 0.569 M_{\odot}$ for SDSS J0349–0059 and $M_{\star} \sim 0.523 M_{\odot}$ for VV 47. If we compare the observed period spacing with the average of the computed period spacings we found $M_{\star} \sim 0.535 M_{\odot}$ for SDSS J0349–0059 and $M_{\star} \sim 0.528 M_{\odot}$ for VV 47. Searching for the best period fit we found, in the case of SDSS J0349–0059, an asteroseismological model with $M_{\star} = 0.542 M_{\odot}$ and $T_{\text{eff}} = 91\,255$ K. For VV 47, we could not find a unique and unambiguous asteroseismological model. Finally, for SDSS J0349–0059, we determined the rotation period of SDSS J0349–0059 as being $P_{\text{rot}} = 1/\Omega \sim 0.407$ days.

Conclusions. The results presented in this work constitute a further step in the study of GW Vir stars through asteroseismology in the frame of fully evolutionary models of PG 1159 stars. In particular, once again it is shown the potential of asteroseismology to derive stellar masses of PG 1159 stars with an unprecedented precision.

Key words. stars: evolution — stars: interiors — stars: oscillations — stars: variables: other (GW Virginis)— white dwarfs

1. Introduction

GW Virginis stars include pulsating stars characterized by several spectral types (see Quirion et al. 2007). Among them, the PG 1159 stars, are very hot H-deficient post-Asymptotic Giant Branch (AGB) stars with surface layers rich in He, C and O (Werner & Herwig 2006; Werner et al. 2014; Werner & Rauch 2015). Pulsating PG 1159 stars exhibit multiperiodic luminosity variations with periods ranging from 5 to 100 minutes, attributable to non-radial pulsation g modes. Some GW Vir are still embedded in a nebula (see the reviews by Winget & Kepler 2008; Fontaine & Brassard 2008; Althaus et al. 2010)¹. PG 1159 stars are thought to be the evolutionary link between Wolf-Rayet type central stars of planetary nebulae and most of the H-deficient white dwarfs (Wesemael et al. 1985; Sion 1986; Althaus et al. 2005). It is generally accepted that these stars

have their origin in a born-again episode induced by a post-AGB He thermal pulse —see Iben et al. (1983); Herwig et al. (1999); Lawlor & MacDonald (2003); Althaus et al. (2005); Miller Bertolami & Althaus (2006) for references.

Notably, considerable observational effort has been invested to study GW Vir stars. Particularly noteworthy are the works of Vauclair et al. (2002) on RX J2117.1+3412, Fu et al. (2007) on PG 0122+200, and Costa et al. (2008) and Costa & Kepler (2008) on PG 1159–035. On the theoretical front, important progress in the numerical modeling of PG 1159 stars (Althaus et al. 2005; Miller Bertolami & Althaus 2006, 2007a,b) has paved the way for unprecedented asteroseismological inferences for the mentioned stars (Córscico et al. 2007a,b, 2008), and also for PG 2131+066, PG 1707+427, NGC 1501, and SDSS J0754+0852 (Córscico et al. 2009a; Kepler et al. 2014). The detailed PG 1159 stellar models of Miller Bertolami & Althaus (2006) were derived from the complete evolutionary history of progenitor stars with different stellar masses and an elaborate treatment of the mixing and extra-mixing processes during the core He burning and born-again phases. It is worth mentioning that these

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¹ For historical reasons, some authors designate Planetary Nebula Nuclei Variable (PNNV) to GW Vir stars still embedded in a nebula, and DOV to GW Vir stars without planetary nebula (Winget & Kepler 2008), a misnomer because no white dwarf of spectral type DO have ever been found to pulsate.

models are characterized by thick helium-rich outer envelopes. The robustness of the H-deficient post-AGB tracks of Miller Bertolami & Althaus (2006) regarding previous evolution of their progenitor stars and the constitutive physics of the remnants have been assessed by Miller Bertolami & Althaus (2007a). The success of these models at explaining the spread in surface chemical composition observed in PG 1159 stars (Miller Bertolami & Althaus 2006), the short born-again times of V4334 Sgr (Miller Bertolami & Althaus 2007b), and the location of the GW Vir instability strip in the $\log T_{\text{eff}} - \log g$ plane (Córscico et al. 2006) renders reliability to the inferences drawn for individual pulsating PG 1159 stars. It is worth mentioning, in the context of nonadiabatic analysis of GW Vir stars, the important theoretical work of Quirion et al. (2004, 2005, 2007, 2012). These studies have shed light on the issues of the presence of variable and non-variable stars in the GW Vir region of the HR diagram, and the existence of the high-gravity red edge of the GW Vir instability domain.

SDSS J0349–0059 is a pulsating PG 1159 star with $T_{\text{eff}} = 90\,000 \pm 900$ K and $\log g = 7.5 \pm 0.01$ (cgs) according to Hügelmeyer et al. (2006), who employed the Data Release 4 of the SDSS Catalogue to estimate these quantities using non-LTE (non-local thermodynamic equilibrium) model atmospheres. Woudt et al. (2012) performed high-speed photometric observations in 2007 and 2009, and found a set of pulsation frequencies in the range of 1038–3323 μHz with amplitudes between 3.5 and 18.6 mmag. The data gathered by Woudt et al. (2012) shows three frequencies closely spaced in the 2007 data, which enables, in principle, to estimate a period rotation for this star.

VV 47 is a PNNV star characterized by $T_{\text{eff}} = 130\,000 \pm 13\,000$ K and $\log g = 7 \pm 0.5$ (Werner & Herwig 2006). Its stellar mass is $M_{\star} = 0.59 M_{\odot}$ according to Werner & Herwig (2006) and $M_{\star} = 0.53 M_{\odot}$ according to the evolutionary tracks created by Miller Bertolami & Althaus (2006). The surface chemical composition of VV 47 is typical of PG 1159 stars: C/He= 1.5 and O/He= 0.4 (Werner & Herwig 2006). VV 47 was first observed as potentially variable by Liebert et al. (1988). Later, it was monitored by Ciardullo & Bond (1996), but no clear variability was found. Finally, González Pérez et al. (2006) were able to confirm the —until then— elusive intrinsic variability of VV 47 for the first time. They found clear evidence that the pulsation spectrum of this star is extremely complex. Indeed, the main peaks of the power spectrum have amplitudes strongly variable between observation seasons, and sometimes they are detected only in a particular run. It is apparent that real periodicities of VV 47 are in the range 131–5682 s. It is important to note that the shortest periods in the observed period spectrum of VV 47 could be associated with the ε -mechanism of mode driving acting at the He burning shell, as suggested by González Pérez et al. (2006). This hypothesis was explored from a theoretical point of view by Córscico et al. (2009b). If this hypothesis were confirmed, this object could be the first known pulsating PG 1159 star undergoing pulsation modes powered by this mechanism.

In this work, we present an adiabatic asteroseismological study of SDSS J0349–0059 and VV 47 aimed at determining the internal structure and evolutionary status of these stars on the basis of the very detailed PG 1159 evolutionary models of Miller Bertolami & Althaus (2006). We emphasize that the results presented in this work could change to some extent if another independent set of PG 1159 evolutionary tracks constructed assuming a different input physics were employed. We compute adiabatic g -mode pulsation periods on PG 1159 evolutionary models with stellar masses ranging from 0.515 to

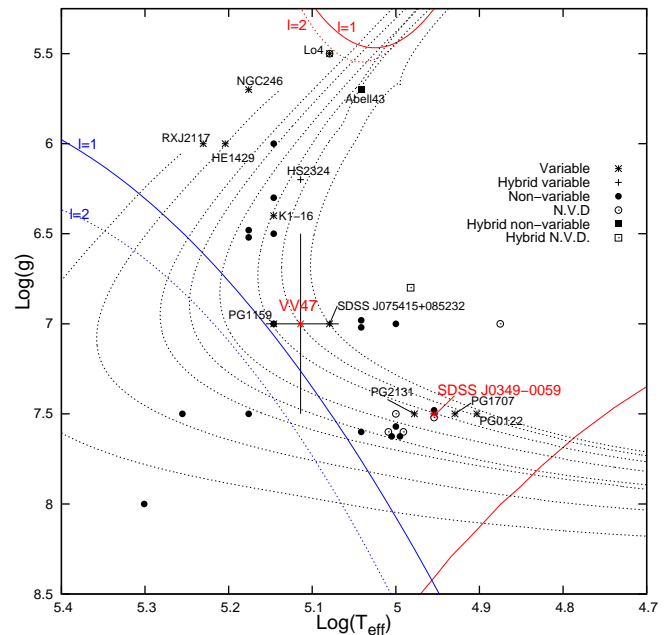


Fig. 1. The PG 1159 (VLTP) evolutionary tracks of Miller Bertolami & Althaus (2006) (from right to left: $M_{\star} = 0.515, 0.530, 0.542, 0.565, 0.589, 0.609, 0.664, 0.741 M_{\odot}$) in the $\log T_{\text{eff}} - \log g$ diagram (thin dotted curves). The location of all known PG 1159 stars (variable, non-variable, and objects with no variability data) is shown, including SDSS J0349–0059 and VV 47 with their uncertainties —the uncertainty values for SDSS J0349–0059 are very small in the adopted scale and they cannot be seen in the plot. The hot blue edges (blue curves) and the low-gravity red edges (upper red curves) of the theoretical GW Vir instability strip for $\ell = 1$ (dashed) and $\ell = 2$ (dotted) modes according to Córscico et al. (2006) are also depicted. The high-gravity red edge due to the competition of residual stellar winds against the gravitational settling of C and O (Quirion et al. 2012) is also included (lower red solid curve).

0.741 M_{\odot} . These models take into account the complete evolution of progenitor stars, through the thermally pulsing AGB phase and born-again episode. A brief summary of the stellar models employed is provided in Sect. 2. We estimate a mean period spacing for both SDSS J0349–0059 and VV 47 (Sect. 3), and then we constrain the stellar masses of these stars by comparing the observed period spacing with the asymptotic period spacing and with the average of the computed period spacings (Sect. 4). In Sect. 5 we employ the individual observed periods to search for a representative seismological model for these stars. In Sect. 6 we estimate the rotation period for SDSS J0349–0059, employing the observed triplet of frequencies. Finally, we close the article with a discussion and summary in Sect. 7.

2. Evolutionary models and numerical tools

The pulsation analysis presented in this work relies on a suite of state-of-the-art stellar models that take into account the complete evolution of the PG 1159 progenitor stars. Specifically, the stellar models were extracted from the evolutionary calculations presented by Althaus et al. (2005), Miller Bertolami & Althaus

(2006), and Córscico et al. (2006), who computed the complete evolution of model star sequences with initial masses on the ZAMS in the range $1 - 3.75 M_{\odot}$. All of the post-AGB evolutionary sequences computed with the LPCODE evolutionary code (Althaus et al. 2005) were followed through the very late thermal pulse (VLTP) and the resulting born-again episode that give rise to the H-deficient, He-, C- and O-rich composition characteristic of PG 1159 stars. The masses of the resulting remnants are 0.530, 0.542, 0.565, 0.589, 0.609, 0.664, and $0.741 M_{\odot}$. An additional evolutionary track of a remnant with a stellar mass of $0.515 M_{\odot}$ is also employed in the case of VV 47. In Fig. 1 the evolutionary tracks employed in this work are shown in the $\log T_{\text{eff}} - \log g$ plane. The blue and red edges of the theoretical GW Vir instability domain according to Córscico et al. (2006) correspond to PG 1159 models with surface fractional abundances (X_i) of ^4He in the range $0.28 - 0.48$, ^{12}C in the range $0.27 - 0.41$ and ^{16}O in the range $0.10 - 0.23$ (see Table 1 of Córscico et al. 2006). For details about the input physics and evolutionary code used, and the numerical simulations performed to obtain the PG 1159 evolutionary sequences employed here, we refer the interested reader to the works by Althaus et al. (2005) and Miller Bertolami & Althaus (2006, 2007a,b).

On the basis of the evolutionary tracks presented in Fig. 1, a value for the spectroscopic mass² of SDSS J0349–0059 and VV 47 can be derived by linear interpolation. In the case of SDSS J0349–0059 we get a stellar mass of $M_{\star} = 0.543 \pm 0.004 M_{\odot}$. It is worth mentioning that this is the first time that a mass determination is done for this star. As for VV 47, it is important to remark that the uncertainties in $\log g$ and $\log T_{\text{eff}}$ are so big than the value derived for the spectroscopic mass is not very accurate. In this case, we get a formal value of $M_{\star} \sim 0.529 M_{\odot}$ (in agreement with the value of $M_{\star} = 0.53 M_{\odot}$ derived by Miller Bertolami & Althaus 2006), although it can be as low as $\sim 0.510 M_{\odot}$ or as high as $\sim 0.609 M_{\odot}$. At this point, it is worth mentioning the interesting work by Quirion et al. (2009), who proposed a novel approach called “non-adiabatic asteroseismology” that leads to a much more precise value of the surface gravity value for VV 47, of $\log g = 6.1 \pm 0.1$. By adopting $T_{\text{eff}} = 130\,000$ K and $\log g = 6.1$, our evolutionary tracks predict a spectroscopic mass of $\sim 0.542 M_{\odot}$.

We computed $\ell = 1, 2$ g-mode adiabatic pulsation periods in the range $80 - 6000$ s with the adiabatic version of the LP-PUL pulsation code (Córscico & Althaus 2006) and the same methods we employed in our previous works³. We analyzed about 4000 PG 1159 models covering a wide range of effective temperatures ($5.4 \gtrsim \log T_{\text{eff}} \gtrsim 4.8$), luminosities ($0 \lesssim \log(L_{\star}/L_{\odot}) \lesssim 4.2$), and stellar masses ($0.515 \leq M_{\star}/M_{\odot} \leq 0.741$).

3. Estimation of a constant period spacing

In the asymptotic limit of high-radial order k , nonradial g modes with the same harmonic degree ℓ are expected to be equally spaced in period (Tassoul 1980):

$$\Delta\Pi_{\ell}^a = \Pi_{k+1,\ell} - \Pi_{k,\ell} = \frac{2\pi^2}{\sqrt{\ell(\ell+1)}} \left[\int_0^{R_{\star}} \frac{N(r)}{r} dr \right]^{-1} \quad (1)$$

² According to the nomenclature widely accepted in the literature, we use the term “spectroscopic mass”, although the term “evolutionary mass” might be more appropriate, because its derivation involves the employment of evolutionary tracks.

³ La Plata Stellar Evolution and Pulsation Research Group (<http://fcaglp.fcaglp.unlp.edu.ar/evolgroup/>)

Table 1. List of the 13 independent frequencies in the 2007 January data of SDSS J0349–0059 from Woudt et al. (2012). The periods that more closely follow a constant period spacing are emphasized with boldface.

Π [s]	Freq. [μHz]	Ampl. [mmag]
963.48 ± 0.37	1037.9 ± 0.4	3.7 ± 0.9
906.37 ± 0.33	1103.3 ± 0.4	3.9 ± 0.9
517.84 ± 0.03	1931.1 ± 0.1	11.3 ± 1.0
504.18 ± 0.05	1983.4 ± 0.2	6.8 ± 0.9
486.40 ± 0.09	2055.9 ± 0.4	4.2 ± 0.9
482.58 ± 0.05	2072.2 ± 0.2	6.6 ± 0.9
465.05 ± 0.09	2150.3 ± 0.4	3.5 ± 0.9
421.48 ± 0.04	2372.6 ± 0.2	6.7 ± 0.9
418.90 ± 0.02	2387.2 ± 0.1	18.6 ± 1.1
416.42 ± 0.02	2401.4 ± 0.1	15.2 ± 1.0
353.79 ± 0.03	2826.5 ± 0.2	7.2 ± 0.9
349.02 ± 0.01	2865.2 ± 0.1	12.1 ± 1.0
300.93 ± 0.03	3323.0 ± 0.3	5.5 ± 0.9

Table 2. List of the 10 independent frequencies in the 2009 December data of SDSS J0349–0059 from Woudt et al. (2012). The periods that best fit a constant period separation are emphasized with boldface.

Π [s]	Freq. [μHz]	Ampl. [mmag]
911.49 ± 0.08	1097.1 ± 0.1	9.1 ± 0.7
680.83 ± 0.14	1468.8 ± 0.3	3.6 ± 0.7
561.83 ± 0.09	1779.9 ± 0.3	4.0 ± 0.7
516.72 ± 0.11	1935.3 ± 0.4	3.3 ± 0.7
511.43 ± 0.08	1955.3 ± 0.3	4.2 ± 0.7
419.55 ± 0.07	2383.5 ± 0.4	3.4 ± 0.7
419.18 ± 0.02	2385.6 ± 0.1	15.2 ± 0.9
412.27 ± 0.03	2425.6 ± 0.2	10.0 ± 0.8
349.06 ± 0.01	2864.8 ± 0.1	9.6 ± 0.8
300.93 ± 0.03	3323.0 ± 0.3	4.3 ± 0.7

where N is the Brunt-Väisälä frequency. In principle, one can compare the asymptotic period spacing or the average of the period spacings computed from a grid of models with different masses and effective temperatures with the mean period spacing exhibited by the star, and then infer the value of the stellar mass (Sect. 4). These methods take full advantage of the fact that the period spacing of pulsating PG 1159 stars depends primarily on the stellar mass, and weakly on the luminosity and the He-rich envelope mass fraction, as it was first recognized by Kawaler (1986, 1987, 1988, 1990) (see also Kawaler & Bradley 1994; Córscico & Althaus 2006). These approaches have been successfully applied in numerous studies of pulsating PG 1159 stars (see, for instance, Córscico et al. 2007a,b, 2008, 2009a). The first step in this process is to obtain (if exists) a mean uniform period separation underlying the observed periods. We searched for a constant period spacing in the data of the stars under analysis by using the Kolmogorov-Smirnov (K-S; see Kawaler 1988), the Inverse Variance (I-V; see O’Donoghue 1994) and the Fourier Transform (F-T; see Handler et al. 1997) significance tests. In the K-S test, the quantity Q is defined as the probability that the observed periods are randomly distributed. Thus, any uniform — or at least systematically non-random — period spacing present in the period spectrum of the star under analysis will appear as a minimum in Q . In the I-V test, a maximum of the inverse variance will indicate the presence of a constant period spacing. Finally, in the F-T test, we calculate the Fourier Transform of a Dirac comb function (created from a set of observed peri-

ods), and then we plot the square of the amplitude of the resulting function in terms of the inverse of the frequency. And once again, a maximum in the square of the amplitude will indicate the presence of a constant period spacing.

3.1. The case of SDSS J0349–0059

We were able to infer an estimate for the period spacing of SDSS J0349–0059 by using the data available in the work of Woudt et al. (2012). In particular, we employed the periods listed in Tables 1 and 2, extracted from Table 2 of Woudt et al. (2012), corresponding to two different observation dates. First, we considered the complete set of periods from both tables. The results of our analysis are shown in Fig. 2. The plot shows evidence of a period spacing at about 23.5 s, but it shows other minima as well, at ~ 17 s in the three tests. Next, we repeated the analysis for several different sets of data in which we discarded one or two periods, following Woudt et al. (2012). We found an unambiguous indication of a constant period spacing of 23.49 s and also a secondary solution of 16.5 s, as it can be seen from Fig. 3, where the periods used are those emphasized with boldface in Tables 1 and 2. By comparing with the $\ell = 1$ period spacing observed in other GW Vir stars: 21.6 s for RXJ2117+3412 (Vauclair et al. 2002); 22.9 s for PG 0122+200 (Fu et al. 2007); 21.4 s for PG 1159–035 (Costa et al. 2008); 21.6 s for PG 2131+066 (Reed et al. 2000); 23.0 s for PG 1707+427 (Kawaler et al. 2004); 22.3 s for NGC 1501 (Bond et al. 1996); and also comparing with the asymptotic period spacing of our PG 1159 models, we can safely identify the period spacing of 23.49 s with $\ell = 1$ modes. We call the attention that we have discarded the second-highest amplitude mode of the 2009 set of modes as well as the fourth-highest amplitude mode of the 2007 set of 13 modes (i.e. $\Pi = 412.27$ s and $\Pi = 517.84$ s, respectively) because they do not match with the determined period spacing associated with $\ell = 1$. We found that they are probably associated with $\ell = 2$ and our models predict their theoretical values to be 410.38 s and 518.10 s, respectively. Regarding the fifth-highest amplitude mode of the 2007 set ($\Pi = 353.79$ s), our models predict that it is associated with $\ell = 1$, but its inclusion or exclusion from the list of the considered periods does not affect the final result.

When we apply the three significance tests to 6 of the discarded periods (i.e. 353.79, 412.27, 482.17, 504.18, 516.18 and 517.84 s), we found a period spacing of 11.6 s, as shown in Fig. 4. At this point, we may wonder whether the spacings at 16.5 s and 11.6 s may be associated with $\ell = 2$ modes. A simple analysis of the asymptotic period spacing helps us to show that this is not the case. Indeed, from Eq. (1) we can see that the relation $\Delta\Pi_{\ell=2}^a = \Delta\Pi_{\ell=1}^a / \sqrt{3}$ between the period spacing corresponding to $\ell = 1$ and $\ell = 2$ holds. So, if the period spacing 23.49 s is associated with $\ell = 1$, then a $\Delta\Pi_{\ell=2}^a \simeq 13.6$ s should be expected for the case of a period separation associated with $\ell = 2$. From this, we conclude that the spacings of 16.5 s and 11.6 s cannot be associated with $\ell = 2$ modes.

We refine the value of the period spacing at 23.49 s by performing a least-squares fit to the set of 15 periods employed in building Fig. 3. We obtain $\Delta\Pi^0 = 23.4904 \pm 0.07741$ s.

In view of the above results, we conclude that there exists strong evidence for a constant period spacing of $\Delta\Pi^0 = 23.49$ s in the pulsation spectrum exhibited by SDSS J0349–0059. This is in perfect agreement with Woudt et al. (2012). We note that the departures from this period spacing could be

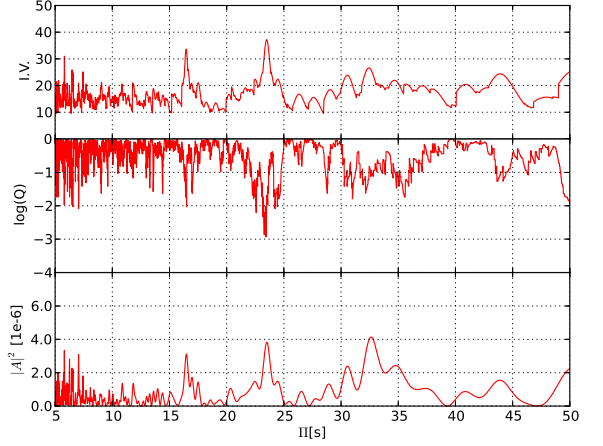


Fig. 2. The I-V (upper panel), the K-S (middle panel) and the F-T significance (bottom panel) tests applied to the period spectrum of SDSS J0349–0059 to search for a constant period spacing. The periods used here are those indicated in Table 1 and 2.

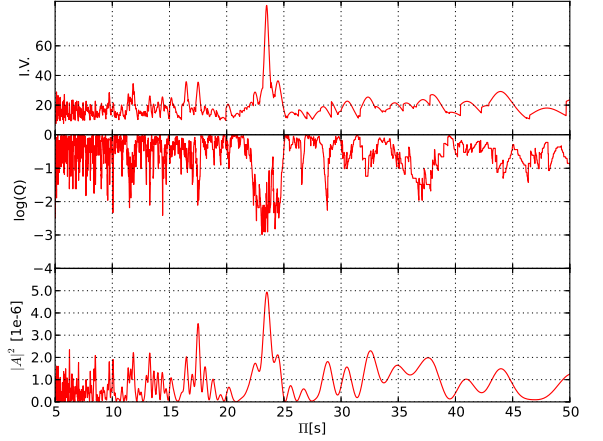


Fig. 3. Same as Fig. 2, but for the 15 selected periods of SDSS J0349–0059 emphasized with boldface in Tables 1 and 2.

associated with the mode-trapping/confinement phenomena (Kawaler & Bradley 1994; Córscico & Althaus 2006).

3.2. The case of VV 47

The analysis presented in this Section is based on the work of González Pérez et al. (2006). Specifically, we use the periods listed in Table 3, extracted from Table 4 of González Pérez et al. (2006), to estimate a period spacing in VV 47.

First, we considered the complete set of periods of Table 3. For simplicity, we adopted a single period of 132.05 s that is the average of the pair of periods at 131.6 s and 132.5 s. The results of our analysis are shown in Fig. 5. In spite of the existence of two minima in $\log Q$ (middle panel), they are not statistically meaningful in the context of the K-S test due to the presence of other several minima with similar significance levels. The F-T based method (lower panel) shows multiple local maxima, and then it is not conclusive. As for the I-V test (upper panel) it does

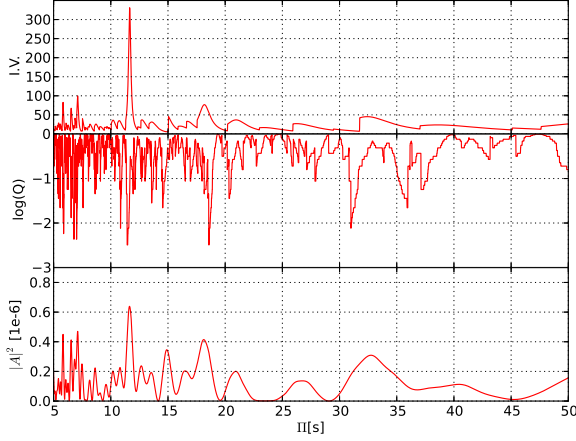


Fig. 4. Same as Fig. 2 but for 6 of the discarded periods of SDSS J0349–0059.

Table 3. The most important peaks in the VV 47 power spectra according to González Pérez et al. (2006). The periods at 240.4, 3521 and 4310 s are present with different amplitudes in different runs. Boldface indicate peaks with best chances to be real.

Π [s]	Freq. [μ Hz]	Power [μ mp]
131.6	7597	2.9
132.5	7550	1.6
153.5	6516	1.6
163.2	6127	1.8
189.2	5286	2.4
211.4	4731	1.6
240.4	4159	3.2
		0.9
261.4	3826	9.4
280.1	3570	2.7
1181	847	1.7
1348	742	1.1
2174	460	3.6
2681	373	2.0
2874	348	1.2
3521	284	2.1
		1.2
4310	232	0.9
		2.0
		5.2
5682	176	1.6

not exhibit any obvious maximum. So, the application of these tests to the complete list of periods does not provide us with any clue about the existence of a constant period spacing in VV 47.

Next, we repeated the above analysis for numerous different sets of periods, ignoring one or more periods from the list, and then we examined the predictions of the three statistical tests. In particular, we found some evidence of the existence of a period spacing of ~ 24 s when the periods at 132.05 s and 153.5 s were departed from the analysis. We also noted that the periods providing the larger discrepancies with that tentative period spacing are 240.4 s, 280.1 s, 3521 s, and 4310 s. Fig. 6 is similar to Fig. 5 but for the situation in which the periods at 132.05 s, 153.5 s, 240.4 s, 280.1 s and 3521 s are not taken into account in the analysis. The plot of the remaining 11 periods shows strong evidence of the presence of a constant period spacing at 24.2 s in the three tests.

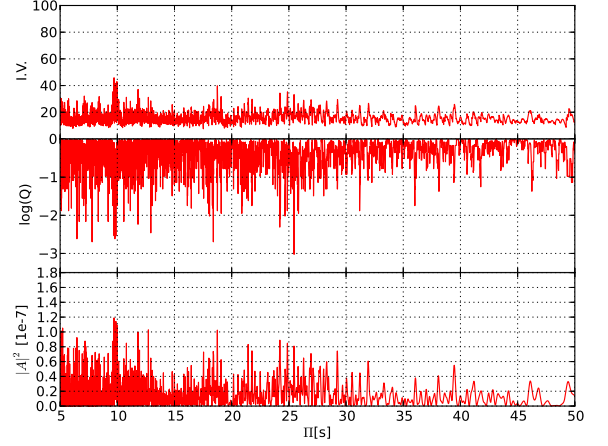


Fig. 5. The I-V (upper panel), the K-S (middle panel) and the F-T (bottom panel) significance tests applied to the period spectrum of VV 47 to search for a constant period spacing. The periods used here are those provided by Table 3, where we considered a single period at 132.05 s instead of the couple of periods at 131.6 s and 132.5 s. No unambiguous signal of a constant period spacing is evident.

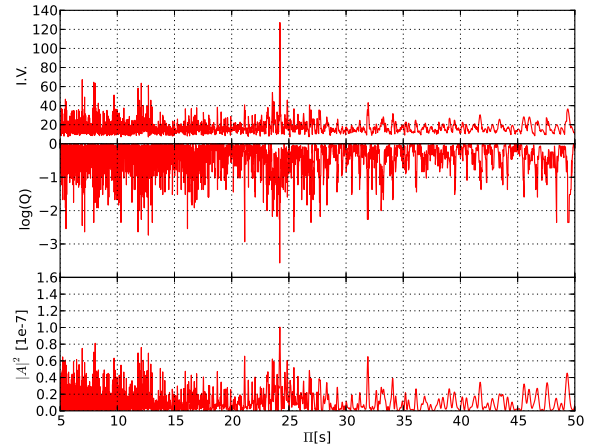


Fig. 6. Same as Fig. 5, but discarding from the analysis the periods at 132.05 s, 153.5 s, 240.4 s, 280.1 s, and 3521 s of VV 47 (see text for details).

To confirm our hypothesized period spacing, we repeated the three tests but this time by discarding also the period at 4310 s, which also presents a significant departure from an equally spaced pattern. We obtained a very strong confirmation for a period spacing of $\Delta\Pi^0 = 24.2$ s by using this subset of 10 periods. This is illustrated in Fig. 7. The agreement between the three methods is excellent. The K-S test indicates that this constant period spacing is significant at a confidence level of $[100 \times (1 - Q)] = 99\%$. The F-T also shows unambiguous evidence of this value. However, the most clear and unambiguous indication of $\Delta\Pi^0 = 24.2$ s is provided by the I-V test. It is also possible to visualize the first harmonic at $\Delta\Pi^0/2 = 12.1$ s shown by all the three tests. Note that the value of the maximum in the inverse variance is more than 5 times larger than in the case displayed in Fig. 6.

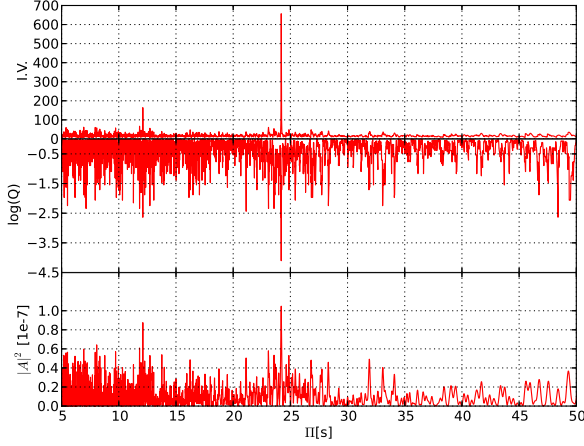


Fig. 7. Same as Fig. 5, but discarding from the analysis the periods at 132.05 s, 153.5 s, 240.4 s, 280.1 s, 3521 s, and 4310 s of VV 47 (see text for details).

We finally mention that the three tests were applied to the five periods with best chance to be real according to González Pérez et al. (2006) (marked with boldface in Table 3). The results did not show any clear indication of the presence of a constant period spacing.

As for SDSS J0349–0059, further confirmation for the period spacing of VV 47 at 24.2 s comes from a least-squares fit to the set of 11 periods employed in constructing Fig. 6. We obtain $\Delta\Pi^0 = 24.2015 \pm 0.03448$ s. With identical arguments than for SDSS J0349–0059 (see Sect. 3.1), we identify this period spacing with $\ell = 1$ g modes.

In view of the above results, we conclude that there exists evidence for a constant period spacing of $\Delta\Pi^0 = 24.2$ s in the pulsation spectrum exhibited by VV 47, and once again, that the departures from this period spacing could be associated with the mode-trapping/confinement phenomena as in the case of SDSS J0349–0059. It is important to note that a constant period spacing at ~ 24 s is by far the largest ever found in pulsating PG 1159 stars. If we assume that this period spacing is associated with $\ell = 1$ modes, a so high $\Delta\Pi^0$ value would imply a rather low total mass value for VV 47, in excellent agreement with the spectroscopic estimation that uses the Miller Bertolami & Althaus (2006) tracks (see the next Section).

4. Mass determination from the observed period spacing

In this section we constrain the stellar mass of SDSS J0349–0059 and VV 47 by comparing the asymptotic period spacing and the average of the computed period spacings with the observed period spacing estimated in the previous Section for each star. As mentioned, these approaches exploit the fact that the period spacing of pulsating PG 1159 stars depends primarily on the stellar mass, and the dependence on the luminosity and the He-rich envelope mass fraction is negligible (Kawaler & Bradley 1994). In the case of VV 47, in order to assess the total mass, we have considered both the high- and low-luminosity regimes of the evolutionary sequences, i.e., before and after the “evolutionary knee” of the tracks (see Fig. 1).

We emphasize that the methods to derive the stellar mass (both spectroscopic and seismic) are not completely independent because the same set of evolutionary models is used in both approaches (see Fontaine & Brassard 2008). Therefore, an eventual agreement between spectroscopic and seismic masses only reflects an internal consistency of the procedure.

4.1. First method: comparing the observed period spacing ($\Delta\Pi_\ell^0$) with the asymptotic period spacing ($\Delta\Pi_\ell^a$)

Fig. 8 shows the asymptotic period spacing $\Delta\Pi_\ell^a$ for $\ell = 1$ modes (calculated according to Eq. 1) as a function of the effective temperature for different stellar masses. Also shown in this diagram are the locations of SDSS J0349–0059, with $\Delta\Pi_{\ell=1}^0 = 23.4904 \pm 0.07741$ s (Sect. 3.1) and $T_{\text{eff}} = 90\,000 \pm 900$ K (Hügelmeier et al. 2006), along with VV 47, with $\Delta\Pi_{\ell=1}^0 = 24.2015 \pm 0.03448$ s (Sect. 3.2) and $T_{\text{eff}} = 130\,000 \pm 13\,000$ K (Werner & Herwig 2006). As can be seen from the Figure, the greater the stellar mass, the smaller the values of the asymptotic period spacing.

By performing a linear interpolation of the theoretical values of $\Delta\Pi_\ell^a$, the comparison between $\Delta\Pi_{\ell=1}^0$ and $\Delta\Pi_{\ell=1}^a$ yields a stellar mass of $M_\star = 0.569^{+0.004}_{-0.002} M_\odot$ for SDSS J0349–0059. Proceeding similarly for VV 47—when there were no points to perform the linear interpolation, we extrapolated the theoretical values of $\Delta\Pi_\ell^a$ —the comparison between $\Delta\Pi_{\ell=1}^0$ and $\Delta\Pi_{\ell=1}^a$ gives a stellar mass of $M_\star = 0.520^{+0.002}_{-0.005} M_\odot$ ($M_\star = 0.526^{+0.007}_{-0.005} M_\odot$) if VV 47 is after (before) the evolutionary knee. Both values are in excellent agreement each other. The errors quoted for VV 47 are admittedly tiny and they are unrealistic because of the large uncertainties in the effective temperature. So, they only represent the internal errors involved in the procedure.

As in our previous works, we emphasize that the derivation of the stellar mass using the asymptotic period spacing may not be entirely reliable in pulsating PG 1159 stars that exhibit short and intermediate periods (see Althaus et al. 2008). This is because the asymptotic predictions are strictly valid in the limit of very high radial order (very long periods) and for chemically homogeneous stellar models, while PG 1159 stars are supposed to be chemically stratified and characterized by strong chemical gradients built up during the progenitor star’s life.

In the next Section we employ another method to infer the stellar mass of PG 1159 stars which, even though it is computationally expensive because requires detailed pulsation computations, it has the advantage of being more realistic.

4.2. Second method: comparing the observed period spacing ($\Delta\Pi_\ell^0$) with the average of the computed period spacings ($\overline{\Delta\Pi_\ell}$)

The average of the computed period spacings is assessed as $\overline{\Delta\Pi_\ell} = (n - 1)^{-1} \sum_k \Delta\Pi_k$, where the “forward” period spacing is defined as $\Delta\Pi_k = \Pi_{k+1} - \Pi_k$ (k being the radial order) and n is the number of theoretical periods within the range of periods observed in the target star. For SDSS J0349–0059, $\Pi_k \in [300, 970]$ s and for VV 47, $\Pi_k \in [160, 5700]$ s.

In Fig. 9 we show the run of the average of the computed period spacings ($\ell = 1$) for SDSS J0349–0059 in terms of the effective temperature for our PG 1159 evolutionary sequences (without the sequence of $M_\star = 0.515 M_\odot$), along with the observed period spacing for SDSS J0349–0059. As can be appreciated from the Figure, the greater the stellar mass, the smaller the values of the average of the computed period spacings. By per-

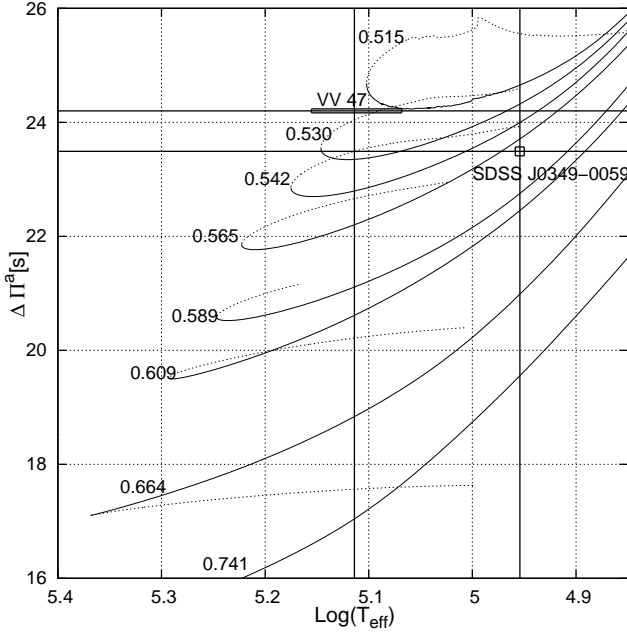


Fig. 8. The dipole ($\ell = 1$) asymptotic period spacing, $\Delta\Pi_\ell^a$, corresponding to each PG 1159 evolutionary sequence considered in this work (dashed and solid curves), in terms of the logarithm of the effective temperature. Numbers along each curve denote the stellar mass (in solar units). Dashed (solid) lines correspond to evolutionary stages before (after) the turning point at the maxima effective temperature of each track (“the evolutionary knee”). The observed period spacing, $\Delta\Pi_{\ell=1}^o$, derived for SDSS J0349–0059 and VV 47 are depicted with horizontal solid lines. The uncertainties in $\Delta\Pi_{\ell=1}^o$ and T_{eff} for each star are also indicated.

forming a linear interpolation of the theoretical values of $\overline{\Delta\Pi}_\ell$, the comparison between $\Delta\Pi_{\ell=1}^o$ and $\overline{\Delta\Pi}_{\ell=1}$ yields a stellar mass of $M_\star = 0.535 \pm 0.004 M_\odot$. This value is smaller than the derived through the asymptotic period spacing, showing that the asymptotic approach overestimates the stellar mass of PG 1159 stars in the case of stars pulsating with short and intermediate periods, which is precisely the situation of SDSS J0349–0059. The value obtained by this approach is more reliable because the method is valid for short, intermediate and long periods, as long as the average of the computed period spacing is evaluated at the right range of periods.

In order to investigate the possibility that the period spacings at ~ 16.5 and ~ 11.6 s found in Sect. 3.1 could be associated with quadrupole ($\ell = 2$) modes, we repeated these calculations, but this time for $\ell = 2$. If $\Delta\Pi_{\ell=2}^o \sim 16.5$ s, then the stellar mass should be much lower than $M_\star = 0.530 M_\odot$. On the other hand, if $\Delta\Pi_{\ell=2}^o \sim 11.6$ s, then the stellar mass should be $\sim 0.741 M_\odot$. In conclusion, both possible period spacing would lead to values of the stellar mass too different than the ones given by the other determinations, in particular, the one derived using the spectroscopic parameters ($M_\star = 0.543 \pm 0.004 M_\odot$, Sect. 2). This analysis, along with the arguments presented in Sect. 3.1 based on the relation between the asymptotic period spacings for $\ell = 1$ and $\ell = 2$, allows to definitely discard these values as probable period spacings associated with $\ell = 2$.

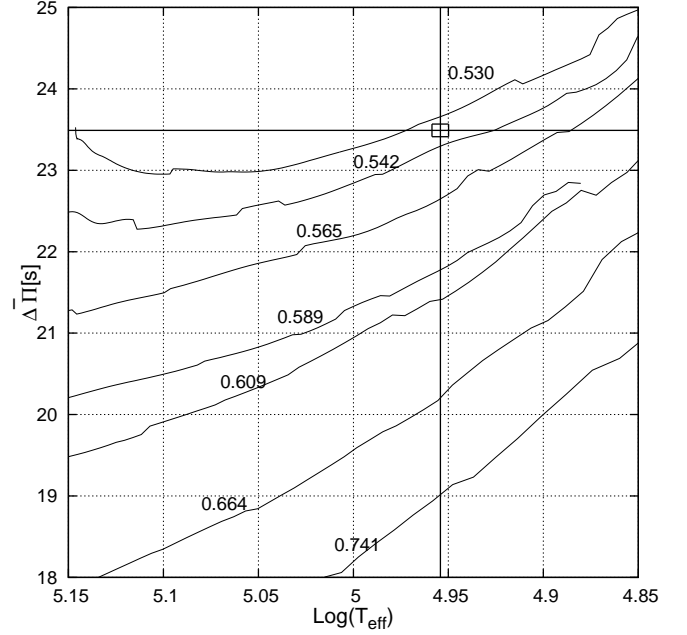


Fig. 9. The dipole average of the computed period spacings, $\overline{\Delta\Pi}_\ell$, computed in the range of periods observed in SDSS J0349–0059, corresponding to each PG 1159 evolutionary sequence considered in this work (solid curves), in terms of the logarithm of the effective temperature. Numbers along each curve denote the stellar mass (in solar units). The observed period spacing, $\Delta\Pi_{\ell=1}^o$, derived for SDSS J0349–0059 is depicted with a horizontal solid line. The uncertainties in $\Delta\Pi_{\ell=1}^o$ and T_{eff} are also indicated.

In the case of VV 47, we depict in Fig. 10 the run of average of the computed period spacings ($\ell = 1$) in terms of the effective temperature for our PG 1159 evolutionary sequences (including $M_\star = 0.515 M_\odot$). The observed period spacing for VV 47 is also shown. By adopting the effective temperature of VV 47 as given by spectroscopy, we found a stellar mass of $M_\star = 0.532_{-0.007}^{+0.004} M_\odot$ if the star is “before the knee”, and $M_\star = 0.524_{-0.001}^{+0.002} M_\odot$ if the star is “after the knee”. These values are very close to those derived from the asymptotic period spacing, as it would be expected on the grounds that this star is pulsating with very long periods, almost in the asymptotic regime.

5. Constraints from the individual observed periods

In this procedure we search for a pulsation model that best matches the *individual* pulsation periods of a given star under study. The goodness of the match between the theoretical pulsation periods (Π_k^T) and the observed individual periods (Π_i^O) is measured by means of a merit function defined as

$$\chi^2(M_\star, T_{\text{eff}}) = \frac{1}{m} \sum_{i=1}^m \min[(\Pi_i^O - \Pi_k^T)^2], \quad (2)$$

where m is the number of observed periods. The PG 1159 model that shows the lowest value of χ^2 , if exists, is adopted as the “best-fit model”. This approach has been used in GW Vir

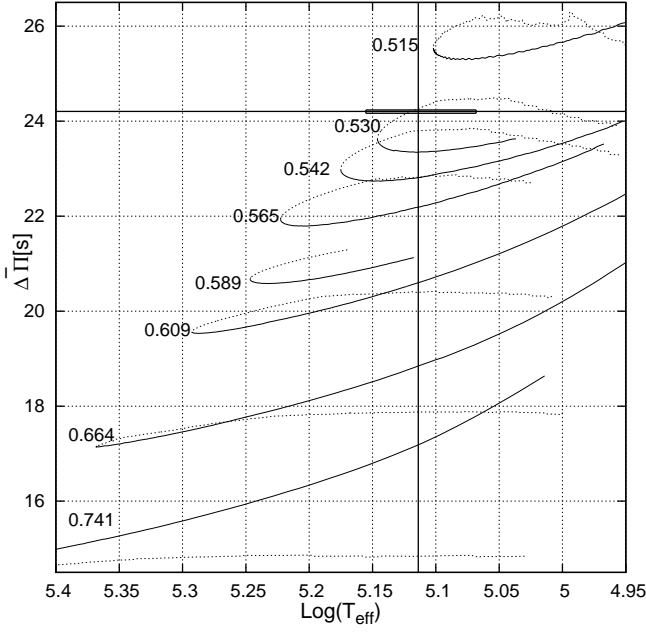


Fig. 10. Same as Fig. 9 but for the case of VV 47.

stars by Córscico et al. (2007a,b, 2008, 2009a) and Kepler et al. (2014). We assess the function $\chi^2 = \chi^2(M_\star, T_{\text{eff}})$ for stellar masses of 0.515, 0.530, 0.542, 0.565, 0.589, 0.609, 0.664, and $0.741 M_\odot$. For the effective temperature we employ a much finer grid ($\Delta T_{\text{eff}} = 10 - 30$ K) which is given by the time step adopted by our evolutionary calculations.

5.1. Searching for the best-fit model for SDSS J0349–0059

We start our analysis assuming that all of the observed periods correspond to $\ell = 1$ modes and consider two different set of observed periods Π_i^0 to compute the quality function given by Eq. (2). We begin by considering the set of periods emphasized in boldface in Tables 1 and 2 but with a difference: the values numerically too close each other were averaged—it is not expected that they correspond to independent modes—resulting at the end in a total of 10 periods to be included in the period fit. Fig. 11(a) shows the quantity $(\chi^2)^{-1}$ in terms of the effective temperature for different stellar masses, taking into account this set of periods. We also include the effective temperature of SDSS J0349–0059 and its uncertainty (vertical lines). As mentioned before, the goodness of the match between the theoretical and the observed periods is measured by the value of χ^2 : the better the period match, the lower the value of χ^2 —in our case, the greater the value of $(\chi^2)^{-1}$. For this case, there is a strong maximum of $(\chi^2)^{-1}$ for a model with $M_\star = 0.609 M_\odot$ and $T_{\text{eff}} \sim 69\,000$ K as it can be seen in Fig. 11(a). But, despite of representing a very good agreement between the observed and the theoretical periods, the effective temperature of this possible solution is unacceptably far from the T_{eff} of SDSS J0349–0059. Another less pronounced maximum correspond to models with: $M_\star = 0.542 M_\odot$ and $T_{\text{eff}} \sim 72\,000$ K, $M_\star = 0.565 M_\odot$ and $T_{\text{eff}} \sim 80\,000$ K, $M_\star = 0.530 M_\odot$ and $T_{\text{eff}} \sim 77\,000$ K, and finally, $M_\star = 0.542 M_\odot$ and $T_{\text{eff}} \sim 91\,000$ K. This last model

is closer to the effective temperature of SDSS J0349–0059 and thus, we adopt this model as the “asteroseismological model”.

Next, we carried out a period fit considering a set of periods from Tables 1 and 2, but adding a set of 6 periods previously discarded (i.e. 353.79 s, 412.27 s, 482.58 s, 504.18 s, 516.72 s, and 517.84 s). Once again, the values numerically too close each other were averaged, so it resulted finally in a set of 11 periods. The outcome of the period fit in this case is displayed in Fig. 11(b). As compared with the results from the first period fit (Fig. 11(a)), the function $(\chi^2)^{-1}$ is characterized by a smoother behavior, although the peak at $\sim 91\,000$ K still remains. As a result, there are not new solutions.

For the adopted asteroseismological model, we can compare the observed and the theoretical periods ($\ell = 1, m = 0$) by computing the absolute period differences $|\delta\Pi| = |\Pi^0 - \Pi^T|$. The results are shown in Table 4. Column 5 of Table 4 shows the value of the linear nonadiabatic growth rate (η), defined as $\eta \equiv -\Im(\sigma)/\Re(\sigma)$, where $\Re(\sigma)$ and $\Im(\sigma)$ are the real and the imaginary part, respectively, of the complex eigenfrequency σ , computed with the nonadiabatic version of the LP-PUL pulsation code (Córscico et al. 2006). A value of $\eta > 0$ ($\eta < 0$) implies an unstable (stable) mode (see column 6 of Table 4). Interestingly enough, most of the periods of the asteroseismological model for SDSS J0349–0059 are associated to unstable modes. Our nonadiabatic computations fails to predict the existence of the unstable modes with periods at ~ 909 s and ~ 963 s observed in the star. The main features of the adopted asteroseismological model for SDSS J0349–0059 are summarized in Table 5.

Table 4. Observed periods compared with theoretical periods corresponding to the the adopted asteroseismological model for SDSS J0349–0059, with $M_\star = 0.542 M_\odot$ and $T_{\text{eff}} = 91\,255$ K ($\ell = 1, m = 0$). Also shown are the radial order k , the absolute period difference, and the nonadiabatic growth rate for each theoretical period.

Π^0 [s]	Π^T [s]	k	$ \delta\Pi $ [s]	η	Remark
300.93	306.45	11	5.52	9.02×10^{-8}	unstable
349.04	354.92	13	5.88	6.48×10^{-7}	unstable
419.04	422.45	16	3.41	4.05×10^{-6}	unstable
465.05	468.23	18	3.18	7.60×10^{-6}	unstable
486.40	490.69	19	4.29	1.19×10^{-5}	unstable
511.43	514.65	20	3.22	9.50×10^{-6}	unstable
561.83	560.07	22	1.76	2.11×10^{-5}	unstable
680.83	674.99	27	5.84	1.74×10^{-5}	unstable
908.93	909.39	37	0.46	-2.79×10^{-4}	stable
963.48	957.00	39	6.48	-7.38×10^{-4}	stable

Table 5. Main characteristics of the adopted asteroseismological model for SDSS J0349–0059.

Quantity	Spectroscopy	Asteroseismology
T_{eff} [K]	$90\,000 \pm 900$	91 255
$\log(g)$	7.5 ± 0.01	7.488
$M_\star [M_\odot]$	—	0.542
$\log(R_\star/R_\odot)$	—	−1.658
$\log(L_\star/L_\odot)$	—	1.475

Next, we considered the possibility that the periods exhibited by the star is actually a mix of $\ell = 1, 2$ g modes. The results are shown in Fig. 12 for the same set of observed periods

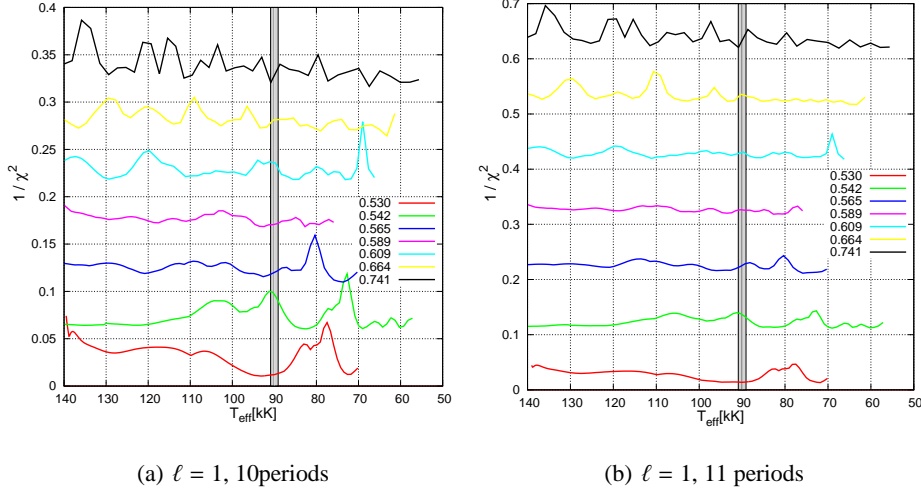


Fig. 11. The inverse of the quality function of the period fit considering $\ell = 1$ versus T_{eff} for SDSS J0349–0059, adopting the sets of 10 (left panel) and 11 (right panel) periods (see text for details). The vertical gray strip depicts the spectroscopic T_{eff} of SDSS J0349–0059 and its uncertainties. For clarity, the curves have been arbitrarily shifted upward (with a step of 0.05 in the left panel and 0.1 in the right panel) except for the lowest curve. The adopted model as the “asteroseismological model” has $M_{\star} = 0.542 M_{\odot}$. [Color figure only available in the electronic version of the article].

considered in Fig. 11(a). In the first case (Fig. 12(a)) there are new possible solutions which in principle can represent good period fits, but the peaks lie well beyond the range allowed by the uncertainties in the effective temperature of SDSS J0349–0059. As for the second set of 11 periods displayed in Fig. 12(b), there are new possible solutions closer to the effective temperature of SDSS J0349–0059, but there is not a definite asteroseismological model. Altogether, since the period fits considering a mix of $\ell = 1$ and $\ell = 2$ modes do not show a clear solution, the results point out into the conclusion that the modes of SDSS J0349–0059 may probably be associated with $\ell = 1$ only.

5.2. Searching for the best-fit model for VV 47

For VV 47, we proceeded analogously to SDSS J0349–0059, but discriminating the cases “before the knee” and “after the knee”. We carry out the procedure considering three different sets of observed periods. The results of the search for the best-fit model for this star “before the knee”, considering only $\ell = 1$ theoretical modes, can be appreciated in Fig. 13. The Fig. 13(a), corresponds to the case of the complete set of periods and it shows no unambiguous asteroseismological model in the range of T_{eff} allowed by the spectroscopy. For the reduced list of 11 periods (see Section 3.2), Fig. 13(b) shows the presence of many local maxima as well, for different masses at several values of the effective temperature, with roughly the same amplitude of $(\chi^2)^{-1}$. However, it is possible to find some models that may constitute good period fits considering the range of the effective temperature, and, at the same time, discarding the models with masses too high (compared to the other mass determinations for this star). These models correspond to a mass of $M_{\star} = 0.530 M_{\odot}$ at $T_{\text{eff}} \sim 124\,500$ K and $T_{\text{eff}} \sim 131\,000$ K. In the third case (Fig. 13(c)) the set corresponds to the 8 longest periods from the complete list (from 1181 s to 5682 s), and once again, there is no unambiguous solution in the range allowed by the effective temperature. However, it is also possible in this

case to discard models, so we may choose tentatively a solution for $M_{\star} = 0.530 M_{\odot}$ and T_{eff} close to the effective temperature of VV 47. It is worth mentioning the presence of a strong peak, present in the three cases considered, corresponding to the model with $M_{\star} = 0.542 M_{\odot}$. However, in that case, the T_{eff} is too low as to be considered a solution, taking into account the constraint given by the spectroscopy. Reversing the argument, the period fit is so good that this might be an indication of a very inaccurate determination of the effective temperature from spectroscopy for this star.

When we analyze the case “after the knee”, we find the results shown in Fig. 14. Again, there are multiple local maxima in the three cases. If we consider only the solutions that lie inside the range given for the uncertainty of the effective temperature, and discard those possible solutions associated with masses too high (as compared with the other mass determinations for this star), we may adopt a possible solution for the mass $M_{\star} = 0.515 M_{\odot}$ at $T_{\text{eff}} \sim 126\,300$ K but this solution does not touch the value of the effective temperature of the star, so it is not completely reliable.

Next, we also consider the possibility that the periods exhibited by VV 47 are a mix of $\ell = 1$ and $\ell = 2$ modes. In Fig. 15 we show the results corresponding to the three set of periods considered for the case in which VV 47 is “before the knee”. For the complete set of periods (Fig. 15(a)) there are several possible equivalent solutions, within the constraint given by the spectroscopy, and consistent with the other determinations of the stellar mass. In the second case (Fig. 15(b)) we can appreciate a possible solution with $M_{\star} = 0.530 M_{\odot}$ which is close to the T_{eff} of VV 47. In the third case (Fig. 15(c)), this possible solution is even more evident, so we conclude that a model with $M_{\star} = 0.530 M_{\odot}$ and $T_{\text{eff}} \sim 130\,000$ K may be considered as a seismological solution, although it is not unique.

Finally, we considered the case in which VV 47 is “after the knee”, and the results are shown in Fig. 16. In the first case, corresponding to the complete set of observed periods (Fig. 16(a)),

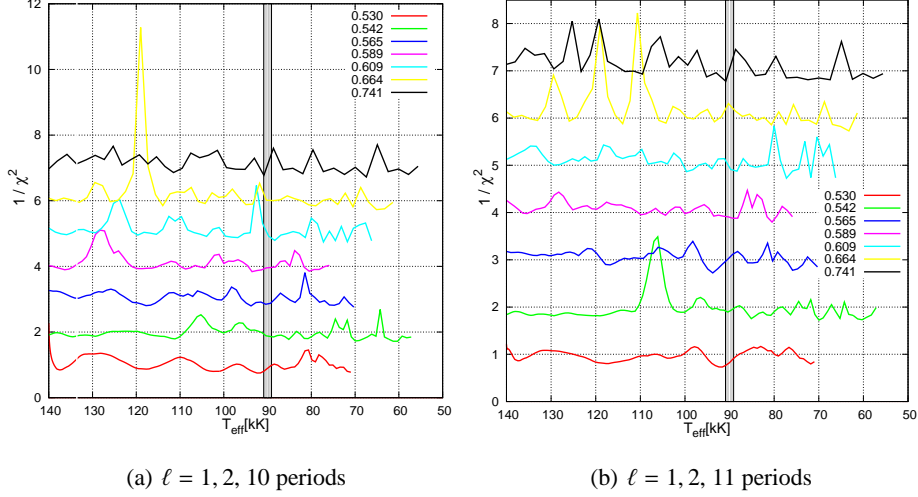


Fig. 12. Same as Fig. 11, but comparing the observed periods with theoretical $\ell = 1$ and $\ell = 2$ periods. In both panels the curves have been arbitrarily shifted upward (with a step of 1) except for the lower curve. [Color figure only available in the electronic version of the article].

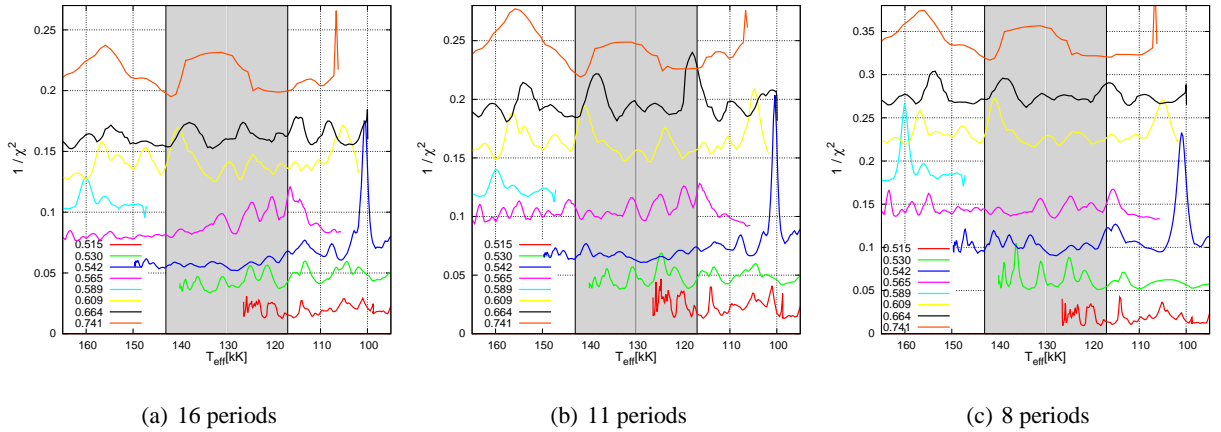


Fig. 13. The inverse of the quality function of the period fit considering $\ell = 1$ modes only in terms of the effective temperature for VV 47 assuming that the star is “before the knee” for three different sets of periods (see text for details). The vertical wide strip in gray depicts the spectroscopic T_{eff} and its uncertainties. The curves have been arbitrarily shifted upward (with a step of 0.02 for the left panel, 0.025 for the middle panel and 0.04 for the right panel) except for the lowest curves. It is possible to adopt a model with $M_{\star} = 0.530 M_{\odot}$ [Color figure only available in the electronic version of the article].

the curves have a rather smooth behavior, and there is not an evident solution. The second case, on the other hand, is displayed in Fig. 16(b). Here, there may be possible solutions within the uncertainty range for T_{eff} that can be adopted as representative models of VV 47 for $M_{\star} = 0.515 M_{\odot}$ at $T_{\text{eff}} \sim 120\,000$ K. In the third case, which is shown in Fig. 16(c), the situation is quite similar to the second case, and we may adopt the same solution. As in the case for $\ell = 1$, this sequence does not reach the value of the effective temperature given for this star, so this solution may not be proper.

We close this Section by emphasizing that, all in all, for VV 47 we have not been able to find a *clear and unambiguous* seismological solution on the basis of our set of PG 1159 evolutionary models. This prevent us to adopt a representa-

tive asteroseismological model for this star (as we did for SDSS J0349–0059), and thus to infer its internal structure.

6. Rotation of SDSS J0349–0059

As noted by Woudt et al. (2012), there is a probable triplet of frequencies in the 2007 data set at $2372.6 \mu\text{Hz}$, $2387.2 \mu\text{Hz}$, and $2401.4 \mu\text{Hz}$ (421.48 s, 418.90 s, and 416.42 s, respectively). This makes feasible a first order analysis of the rotational splitting in order to derive an estimate the rotation period, $P_{\text{rot}} = 1/\Omega$, of SDSS J0349–0059.

In absence of rotation, each eigenfrequency of a nonradially pulsating star is $(2\ell + 1)$ -fold degenerate. Under the assumption of slow rotation, which is the case for most of white dwarfs and

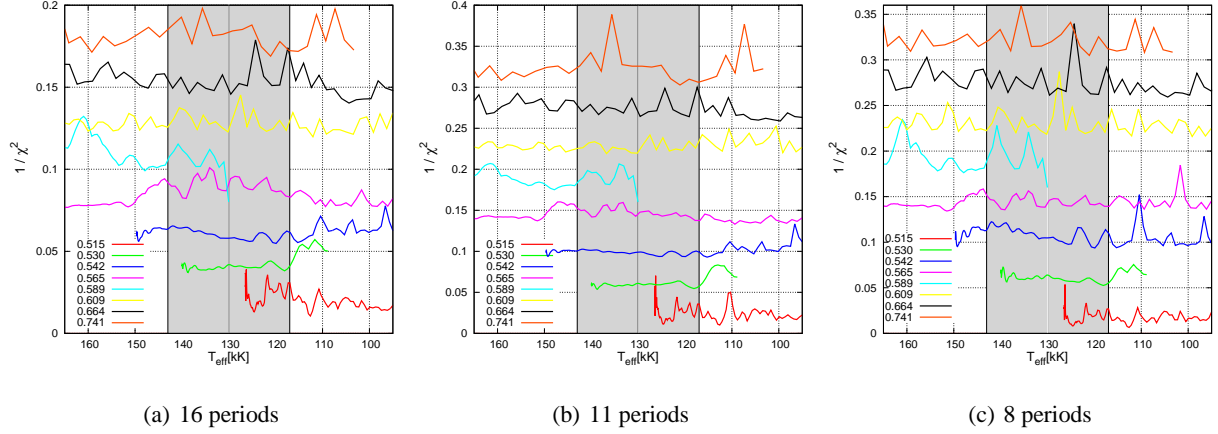


Fig. 14. Same as Fig. 13, but for the case in which VV 47 is evolving “after the knee”. The curves have been arbitrarily shifted upward (with a step of 0.02 for the left panel and 0.04 for the middle and right panel) except for the lower curve.

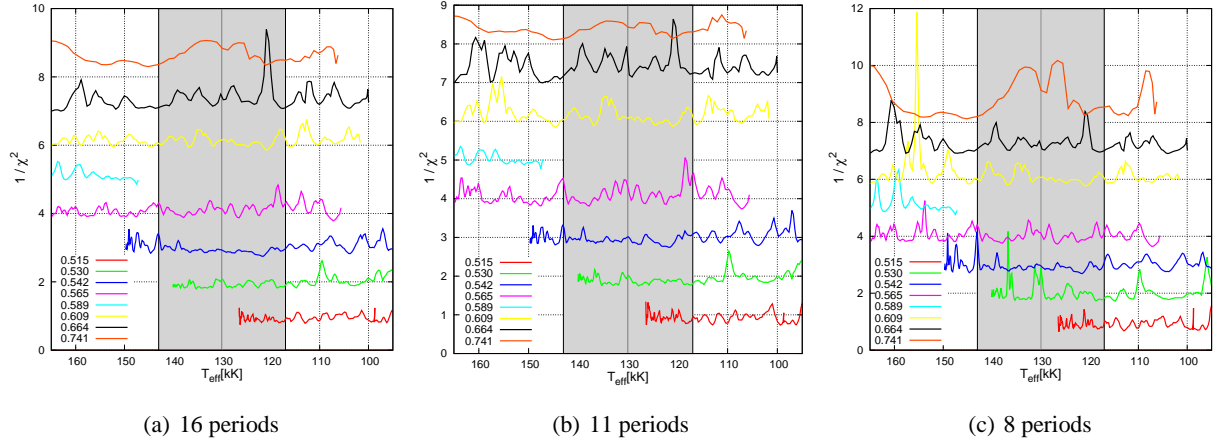


Fig. 15. The inverse of the quality function of the period fit considering a mix of $\ell = 1, 2$ modes in terms of the effective temperature for VV 47 “before the knee” for three different set of periods (see text for details). The vertical gray strip depicts the spectroscopic T_{eff} and its uncertainties. The curves have been arbitrarily shifted upward (with a step of 1) except for the lowest curve. It is possible to adopt a model with $M_{\star} = 0.530 M_{\odot}$ [Color figure only available in the electronic version of the article].

pre-white dwarfs (Fontaine & Brassard 2008; Charpinet et al. 2009; Althaus et al. 2010), the perturbation theory can be applied to the first order. It turns out that the degeneracy of frequencies is lifted, and each component of the resulting multiplet can be calculated as: $\sigma_{k,\ell,m}(\Omega) = \sigma_{k,\ell}(\Omega = 0) + \delta\sigma_{k,\ell,m}$ (Cowling & Newing 1949). If rotation is rigid ($\Omega = \text{constant}$) the first-order corrections to the frequency can be expressed as: $\delta\sigma_{k,\ell,m} = -m\Omega(1 - C_{k,\ell})$, where $m = 0, \pm 1, \dots, \pm\ell$, and $C_{k,\ell}$ are coefficients that depend on the details of the stellar structure and the eigenfunctions, and can be obtained in the non-rotating case. When an asteroseismological model is found for the star under study, such coefficients are available (i.e. they result from the pulsation calculations) and so, it is our case. For SDSS J0349–0059, the asteroseismological model is characterized by $T_{\text{eff}} = 91\,255$ K and $M_{\star} = 0.542 M_{\odot}$ (Table 5). If we associate the periods of SDSS J0349–0059 at 416.42 s and 421.48 s to the components of a rotational triplet with $m = +1$ and $m = -1$ respectively, and assume the period

418.90 s as the central component ($m = 0$) (see Table 1), then $\Delta\sigma = \sigma(m = +1) - \sigma(m = -1) = 28.8 \mu\text{Hz}$ is the frequency spacing between the extreme components of the triplet. Thus, $\delta\sigma = \Delta\sigma/2 = 14.4 \mu\text{Hz}$. The corresponding value for the coefficient $C_{k,\ell}$ is 0.4936, corresponding to the theoretical period closest to the observed period with $m = 0$ of the asteroseismological model for SDSS J0349–0059. Then, $\Omega = 28.4360 \mu\text{Hz}$, leading to a rotation period of $P_{\text{rot}} = 1/\Omega = 0.407$ d.

7. Summary and conclusions

In this work, we presented a detailed asteroseismological study of the pulsating PG 1159 stars SDSS J0349–0059 and VV 47, aimed at determining the internal structure and evolutionary stage of these pulsating stars. Our analysis is based on the fully evolutionary PG 1159 models of Althaus et al. (2005), Miller Bertolami & Althaus (2006) and Córscico et al. (2006). The observational data employed for this study was based on the

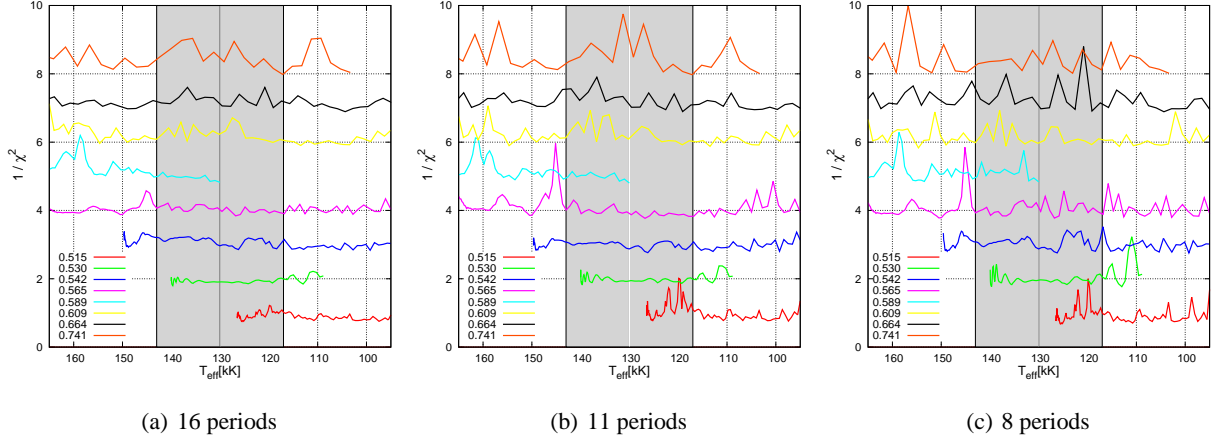


Fig. 16. Same as Fig. 15, but for the case in which VV 47 is “after the knee”.

observed periods reported in the works of Woudt et al. (2012) for SDSS J0349–0059, and González Pérez et al. (2006) for VV 47. Employing the spectroscopic data from Hügelmeier et al. (2006) for SDSS J0349–0059 and Werner & Herwig (2006) for VV 47, we inferred a value for the spectroscopic mass of both stars. The results are $M_{\star} = 0.543 M_{\odot}$ for SDSS J0349–0059 and $M_{\star} = 0.529 M_{\odot}$ for VV 47 (see Fig. 1).

Next, we determined the observed period spacing for both stars, employing three different and independent tests. We found $\Delta\Pi_{\ell=1}^{\circ} = 23.4904 \pm 0.07741$ s for SDSS J0349–0059 and $\Delta\Pi_{\ell=1}^{\circ} = 24.2015 \pm 0.03448$ s for VV 47 (see Sect. 3). Then, making use of the strong dependence of the period spacing of pulsating PG 1159 stars on the stellar mass, we derived the mass for both stars under study. First, it was achieved by comparing the observed period spacing with the asymptotic period spacing of our models (which is an inexpensive approach, since it that does not involve pulsation computations). We obtained $M_{\star} = 0.569^{+0.004}_{-0.002} M_{\odot}$ for SDSS J0349–0059. For VV 47, we derived $M_{\star} = 0.526^{+0.007}_{-0.005} M_{\odot}$ if the star is “before the knee”, and $M_{\star} = 0.520^{+0.002}_{-0.005} M_{\odot}$ for VV 47, if instead it is “after the knee” (see Sect. 4.1). A second estimate of M_{\star} , based on the comparison of the observed period spacings with the average of the computed period spacings (an approach that requires of detailed period computations), gives $M_{\star} = 0.535 \pm 0.004 M_{\odot}$ for SDSS J0349–0059, and $M_{\star} = 0.532^{+0.004}_{-0.007} M_{\odot}$ (“before the knee”) and $M_{\star} = 0.524^{+0.002}_{-0.001} M_{\odot}$ (“after the knee”) for VV 47 (see Sect. 4.2). A third determination was achieved by carrying out period-to-period fits, consisting in searching for models that best reproduce the individual observed periods of each star. The period fits were made on a grid of PG 1159 models with a fine resolution in stellar mass and a much finer grid in effective temperature and considering g modes with $\ell = 1$ and $\ell = 2$. In the case of SDSS J0349–0059, we were able to find an asteroseismological model with $M_{\star} = 0.542 M_{\odot}$ and $T_{\text{eff}} = 91\,255$ K (for $\ell = 1$ g modes) based on the constraint given by the spectroscopy (see 5.1). The search for a period fit for modes $\ell = 1$ and $\ell = 2$ simultaneously did not result in an asteroseismological solution, thus indicating that the periods exhibited by this star are associated only to $\ell = 1$ modes. In the case of VV 47 there is not a clear and unambiguous solution (see Sect. 5.2), which unfortunately prevents us to present a representative asteroseismological model for this star and to extract seismic information

of its internal structure. Finally, for SDSS J0349–0059, once we adopt the model with $M_{\star} = 0.542 M_{\odot}$ as the asteroseismological model for this star, we were able to determine the rotation period employing the observed triplet of frequencies associated with the period 418.90 s ($m = 0$) (see Sect. 6). We found a rotation period of $P_{\text{rot}} = 1/\Omega = 0.407$ d.

In Table 6 we show a compilation of the mass determinations carried out for the most studied pulsating PG 1159 stars, analyzed on the basis of our set of fully evolutionary models. For the case of VV 47, the values quoted for the mass result from averaging the two estimates obtained in this paper considering that the star is evolving before and after the evolutionary knee.

In summary, we were able to find an excellent agreement between our estimates for the stellar mass of both SDSS J0349–0059 and VV 47, something that shows the great internal consistency of our analysis. The fact that we were able to find an asteroseismological model for SDSS J0349–0059 implies that we have additional information about this star, such as the stellar radius, luminosity and gravity (see Table 5). On the other hand, as expected, the determination of the rotation period shows a good agreement with the one estimated by Woudt et al. (2012) of $P_{\text{rot}} = 1/\Omega = 0.40 \pm 0.01$ d, and it is also in line with the values determined for other white dwarf and pre-white dwarf stars (see Fontaine & Brassard (2008), Table 4). This result reinforces the belief that pre-white dwarf stars are slow rotators. We mention that it would be valuable to repeat these calculations using an independent set of evolutionary tracks of PG 1159 stars to add robustness to our results. This is beyond the scope of this paper.

The present paper constitutes a further step in a series of studies done by our group aiming at studying the internal structure and evolution status of pulsating PG 1159-type stars through the tools of asteroseismology. From the results presented in this paper for the pulsating PG 1159 stars SDSS J0349–0059 and VV 47, it is evident once again the power of this approach, in particular for determining the stellar mass with an unprecedented precision.

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Table 6. Stellar masses for all of the intensively studied pulsating PG 1159 stars. All masses are in solar units.

Star	$\Delta\Pi_\ell^a$	$\overline{\Delta\Pi}_\ell$	Period fit	Pulsations (other works)	Spectroscopy
NGC 1501	0.571 ^a	0.576 ^a	—	0.55 ^j (asymptotic analysis)	0.56
RX J2117.1+3412	0.568 ^b	0.560 ^b	0.565 ^b	0.56 ^h (asymptotic analysis)	0.72
PG 1159–035	0.577–0.585 ^d	0.561 ^d	0.565 ^d	0.59 ⁱ (asymptotic analysis)	0.54
PG 2131+066	0.627 ^a	0.578 ^a	0.589 ^a	0.61 ^c (period fit)	0.55
PG 1707+427	0.597 ^a	0.566 ^a	0.542 ^a	0.57 ^g (asymptotic analysis)	0.53
PG 0122+200	0.625 ^c	0.567 ^c	0.566 ^c	0.69 ^f (asymptotic analysis)	0.53
SDSS J0754+0852	—	—	0.556 ^k	—	0.52
SDSS J0349–0059	0.569 ^l	0.535 ^l	0.542 ^l	—	0.54
VV 47	0.523 ^l	0.528 ^l	0.523 ^l	—	0.51–0.61

Notes: ^aCórscico et al. (2009a). ^bCórscico et al. (2007a). ^cCórscico et al. (2007b). ^dCórscico et al. (2008). ^eReed et al. (2000). ^fFu et al. (2007). ^gKawaler et al. (2004). ^hVauclair et al. (2002). ⁱCosta et al. (2008). ^jBond et al. (1996). ^kKepler et al. (2014). ^lThis work.

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